2886. [2003 : 468] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

If a, b, c are positive real numbers such that $abc = 1$, prove that

$$ab^2 + bc^2 + ca^2 \;\geq\; ab + bc + ca\,.$$

I. *Nearly identical solutions Chip Curtis, Missouri Southern State College, Joplin, MO, USA; Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA; Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; and Li Zhou, Polk Community College, Winter Haven, FL, USA.*

By the AM–GM Inequality, $\dfrac{ab^2+2bc^2}{3} \geq \sqrt[3]{(ab^2)(bc^2)^2} = bc$, and similarly, $\dfrac{bc^2+2ca^2}{3} \geq ca$ and $\dfrac{ca^2+2ab^2}{3} \geq ab$. Adding the three inequalities completes the proof.

II. *Solution by Christopher J. Bradley, Bristol, UK.*

Since $abc = 1$, the inequality is equivalent to

$$\frac{b}{c}+\frac{c}{a}+\frac{a}{b} \;\geq\; \frac{1}{c}+\frac{1}{a}+\frac{1}{b} \tag{1}$$

Applying the Cauchy–Schwarz Inequality to the vectors $\left[\sqrt{\frac{b}{c}}, \sqrt{\frac{c}{a}}, \sqrt{\frac{a}{b}}\right]$ and $\left[\frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{a}}\right]$, we have $\left(\frac{b}{c}+\frac{c}{a}+\frac{a}{b}\right)\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right) \geq \left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)^2$, from which (1) follows.

III. *Similar solutions by Arkady Alt, San Jose, CA, USA; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Joe Howard, Portales, NM, USA; and Titu Zvonaru, Bucharest, Romania.*

Since $abc = 1$, there are positive real numbers x, y, z such that $a = \frac{x}{y}$, $b = \frac{y}{z}$, and $c = \frac{z}{x}$. The given inequality is then equivalent to

$$\begin{aligned} \frac{xy}{z^2}+\frac{yz}{x^2}+\frac{zx}{y^2} \;&\geq\; \frac{x}{z}+\frac{y}{x}+\frac{z}{y}\,, \\ \text{or}\qquad x^3y^3+y^3z^3+z^3x^3 \;&\geq\; x^3y^2z+xy^3z^2+x^2yz^3\,. \end{aligned} \tag{2}$$

Inequality (2) follows from Muirhead's Theorem on majorization since the vector $[3,3,0]$ majorizes the vector $[3,2,1]$. Note that equality holds if and only if $x = y = z$; that is, if and only if $a = b = c = 1$. Alternately, the AM–GM Inequality could be applied to obtain

$$x^3y^3 + 2y^3z^3 \;\geq\; 3\sqrt[3]{(x^3y^3)(y^3z^3)^2} \;=\; 3xy^3z^2\,.$$

Similarly, $y^3z^3 + 2z^3x^3 \geq 3x^2yz^3$ and $z^3x^3 + 2x^3y^3 \geq 3x^3y^2z$.

Adding these three inequalities, (2) follows.